Lecture 4: CAPM

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FINM 36700: Portfolio Management

Outline

The CAPM

The most famous Linear Factor Model is the Capital Asset Pricing Model (CAPM).

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^m\right] \tag{1}$$

$$\beta^{i,m} \equiv \frac{\operatorname{cov}\left(\tilde{r}^i, \tilde{r}^m\right)}{\operatorname{var}\left(\tilde{r}^m\right)}$$

where $\tilde{r}^m$ denotes the return on the entire market portfolio, meaning a portfolio that is value-weighted to every asset in the market.

The market portfolio

The CAPM identifies the **market portfolio** as the tangency portfolio.

- The market portfolio is the value-weighted portfolio of all available assets.
- It should include every type of asset, including non-traded assets.
- In practice, a broad equity index is typically used.

Explaining expected returns

The CAPM is about expected returns:

- The expected return of any asset is given as a function of two market statistics: the risk-free rate and the market risk premium.
- The coefficient is determined by a regression. If β were a free parameter, then this theory would be vacuous.
- In this form, the theory does not say anything about how the risk-free rate or market risk premium are given.
- Thus, it is a relative pricing formula.

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Deriving the CAPM

If returns have a joint normal distribution...

1. The mean and variance of returns are sufficient statistics for the return distribution.
2. Thus, every investor holds a portfolio on the $\tilde{M}$V frontier.
3. Everyone holds a combination of the tangency portfolio and the risk-free rate.
4. Then aggregating across investors, the market portfolio of all investments is equal to the tangency portfolio.

Deriving CAPM by investor preferences

Even if returns are not normally distributed, the CAPM would hold if investors only care about mean and variance of return.

- This is another way of assuming all investors choose MV portfolios.
- But now it is not because mean and variance are sufficient statistics of the return distribution, but rather that they are sufficient statistics of investor objectives.
- So one derivation of the CAPM is about return distribution, while the other is about investor behavior.

CAPM assumptions and asset classes

But if we assume normally distributed and iid. returns...

- Application is almost exclusively for equities.
- The CAPM is often not even tried on derivative securities, or even debt securities.

The CAPM decomposition of risk premium

The CAPM says that the risk premium of any asset is proportional to the market risk premium.

$$\mathbb{E}\left[\tilde{r}^{i}\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^{m}\right] \tag{2}$$

The **risk premium** of an asset is defined as the **expected excess return** of that asset.

- The scale of proportionality is given by a measure of risk—the market beta of asset i.
- What would a negative beta indicate?

Beta as the only priced risk

Equation (2) says that market beta is the **only** risk associated with higher average returns.

- No other characteristics of asset returns command a higher risk premium from investors.
- Beyond how it affects market beta, CAPM says volatility, skewness, other covariances do not matter for determining risk premia.

Return variance decomposition

The CAPM implies a clear relation between volatility of returns and risk premia.

$$\tilde{r}^i_t = \beta^{i,m}\tilde{r}^m_t + \epsilon_t$$

Take the variance of both sides of the equation to get

$$\sigma_i^2 = \underbrace{\left(\beta^{i,m}\right)^2 (\sigma^m)^2}_{\text{systematic}} + \underbrace{\sigma_\epsilon^2}_{\text{idiosyncratic}}$$

So CAPM implies...

- The variance of an asset's return is made up of a systematic (or market) portion and an idiosyncratic portion.
- Only the former risk is priced.

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Proportional risk premium

To appreciate how idiosyncratic risk does not increase return, consider the following calculations for expected returns.

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^m\right]$$

- Using the definition of $\beta^{i,m}$,

$$\frac{\mathbb{E}\left[\tilde{r}^i\right]}{\sigma^i} = \left(\rho^{i,m}\right)\frac{\mathbb{E}\left[\tilde{r}^m\right]}{\sigma^m} \tag{3}$$

where $\rho^{i,m}$ denotes $\text{corr}\left(\tilde{r}^m, \tilde{r}^i\right)$.

The CAPM and Sharpe-Ratios

Using the definition of the Sharpe ratio in (3), we have

$$\mathrm{SR}^i = \left(\rho^{i,m}\right)\mathrm{SR}^m$$

- The Sharpe ratio earned on an asset depends only on the correlation between the asset return and the market.
- A security with large idiosyncratic risk, σ^2_ϵ, will have lower $\rho^{i,m}$ which implies a lower Sharpe Ratio.
- Thus, risk premia are determined only by systematic risk.

Treynor's Ratio

If CAPM does not hold, then Treynor's Measure is not capturing all priced risk.

$$\text{Treynor Ratio} = \frac{\mathbb{E}\left[\tilde{r}^i\right]}{\beta^{i,m}}$$

If the CAPM does hold, then what do we know about Treynor Ratios?

Outline

The CAPM

Testing

Fama-MacBeth

CAPM and realized returns

The CAPM implies that expected returns for any security are

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\,\mathbb{E}\left[\tilde{r}^m\right]$$

This implies that realized returns can be written as

$$\tilde{r}^i_t = \beta^{i,m}\ \tilde{r}^m_t + \epsilon_t \tag{4}$$

where ϵ_t is **not** assumed to be normal, but

$$\mathbb{E}\left[\epsilon\right] = 0$$

Of course, taking expectations of both sides we arrive back at the expected-return formulation.

Testing the CAPM on an asset

Using any asset return i, we can test the CAPM.

- Run a time-series regression of excess returns i on the excess market return.
- Regression for asset i, across multiple data points t:

$$\tilde{r}_t^i = \alpha^i + \beta^{i,m}\, \tilde{r}_t^m + \epsilon_t^i$$

  Estimate α and β.

- The CAPM implies $\alpha^i = 0$.

Testing the CAPM on a group of assets

Can run a CAPM regression on various assets, to get various estimates α^i.

- CAPM claims every single α^i should be zero.
- A joint-test on the α^i should not be able to reject that all α^i are jointly zero.

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CAPM and realized returns

CAPM explains variation in $\mathbb{E}\left[\tilde{r}^i\right]$ across assets—NOT variation in $\tilde{r}^i$ across time!

$$\tilde{r}^i_t = \alpha^i + \beta^{i,m}\ \tilde{r}^m_t + \epsilon_t$$

- The CAPM does not say anything about the size of ϵ_t.
- Even if the CAPM were exactly true, it would not imply anything about the R^2 of the above regression, because σ_ϵ may be large.

CAPM as practical model

For many years, the CAPM was the primary model in finance.

- In many early tests, it performed quite well.
- Some statistical error could be attributed to difficulties in testing.
- For instance, the market return in the CAPM refers to the return on all assets—not just an equity index. (Roll critique.)
- Further, working with short series of volatile returns leads to considerable statistical uncertainty.

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Industry portfolios

A famous test for the CAPM is a collection of industry portfolios.

- Stocks are sorted into portfolios such as manufacturing, telecom, healthcare, etc.
- Again, variation in mean returns is fine if it is accompanied by variation in market beta.

Industry portfolios: beta and returns

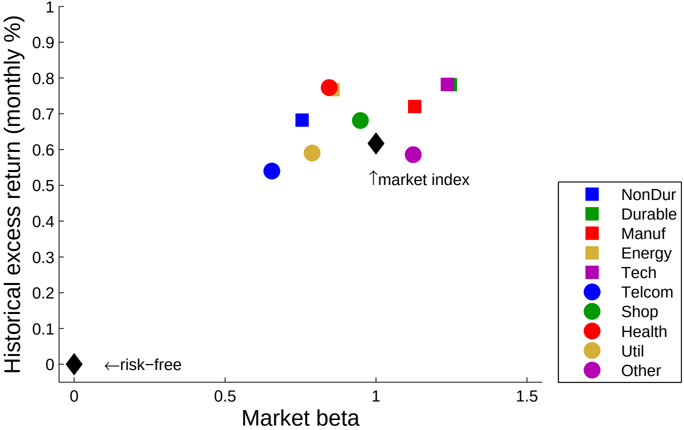

Figure: Data Source: Ken French. Monthly 1926-2011.

Evidence for CAPM?

The plot of industry portfolios shows monthly risk premia from about 0.5% to 0.8%.

- Still, there is substantial spread in betas, and the correlation seems to be positive.
- Note that the risk-free rate and market index are both plotted (black diamonds.)
- Note that the markers for the “Health” and “Tech” portfolio cover up most of the markers for “Energy” and “Durables”.

CAPM-implied relation between beta and returns

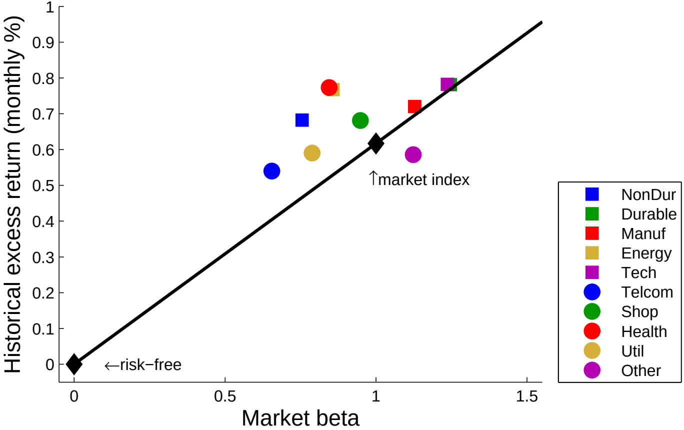

Figure: Data Source: Ken French. Monthly 1926-2011.

CAPM and risk premium

CAPM can be separated into two statements:

- Risk premia are proportional to market beta:

$$\mathbb{E}\left[\tilde{r}^i\right] = \beta^{i,m}\lambda_m \tag{5}$$

- The proportionality is equal to market risk premium:

$$\lambda_m = \mathbb{E}\left[\tilde{r}^m\right] \tag{6}$$

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The risk-return tradeoff

The parameter λ_m is particularly important.

- It represents the amount of risk premium an asset gets per unit of market beta.
- Thus, can divide risk premium, into quantity of risk, $\beta^{i,m}$, multiplied by **price of risk**, λ_m .
- λ_m is also the slope of the **Security Market Line** (SML), which is the line plotted in slide 24.

Cross-sectional test of the CAPM

We can run a cross-sectional regression to test implications (5) and (6).

$$\mathbb{E}\left[\tilde{r}^i\right] = \underbrace{\eta}_{\alpha} + \underbrace{\beta^{i,m}}_{x^i}\underbrace{\lambda_m}_{\beta^i} + \underbrace{\upsilon^i}_{\epsilon^i}$$

- The data on the left side is a list of mean returns on assets, $\mathbb{E}\left[\tilde{r}^i\right]$.
- The data on the right side is a list of asset betas: $\beta^{i,m}$ for each asset i.
- The regression parameters are η and λ_m.
- The regression errors are υ^i.

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CAPM implications in the cross-section

$$\mathbb{E}\left[\tilde{r}^i\right] = \eta + \beta^{i,m}\lambda_m + \upsilon^i$$

- CAPM statement (5) implies the R^2 of the cross-sectional regression is 100%.
$$\upsilon^i = 0, \forall i$$
- CAPM statement (6) implies the cross-sectional regression parameters are:
$$\eta = 0, \qquad \lambda_m = \mathbb{E}\left[\tilde{r}^m\right]$$
- That is, the SML goes through zero and the market return. (See slide 24.)

Estimating the cross-sectional CAPM equation

Estimation of the cross-sectional equation on industry portfolios shows:

- The estimated slope, λ_m is too small relative to the full CAPM theory.
- The SML line doesn't start at zero, $\eta > 0$.

This is a well-known fact. (But only a puzzle if you really believe the CAPM!)

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Unrestricted SML for industry portfolios;

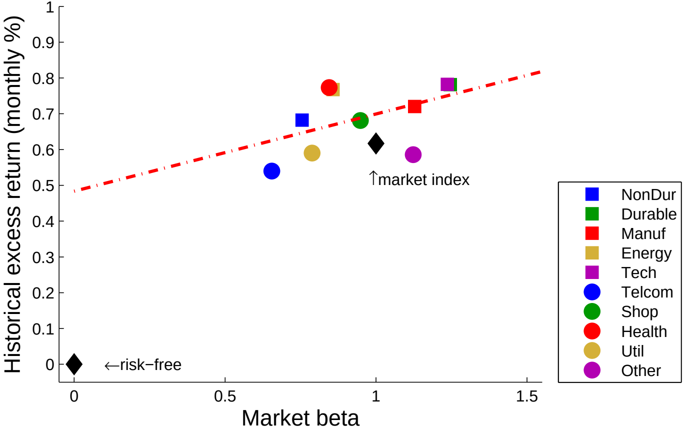

Figure: Data Source: Ken French. Monthly 1926-2011.

Risk-reward tradeoff is too flat relative to CAPM

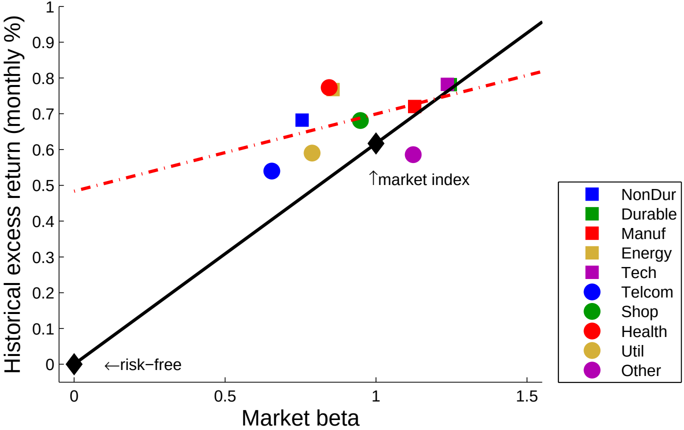

Figure: Data Source: Ken French. Monthly 1926-2011.

Trading on the security market line

Suppose one believes the CAPM: market beta completely describes (priced) risk.

- Relatively small λ_m in estimation implies that there is little difference in mean excess returns even as risk $\left(\beta^{i,m}\right)$ varies.
- A trading strategy would then be to bet against beta: go long small-beta assets and short large-beta assets.
- Frazzini and Pedersen (2011) have an interesting paper on this.

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Fama-MacBeth

Time-varying beta

We want to allow for beta to vary over time.

$$\tilde{r}_t^i = \alpha^i + \beta_t^{i,z} z_t + \epsilon_t^i$$

So far, we have been estimating unconditional β

$$\tilde{r}_t^i = \alpha^i + \beta^{i,z} z_t + \epsilon_t^i$$

Must choose a model for how β changes over time.

- Consider stochastic vol models above.
- Often see estimates of β_t using rolling window of data. 5 years?
- Can use GARCH, other models to capture nonlinear impact.

Fama-Macbeth estimates

The Fama-Macbeth procedure is widely used to deal with time-varying betas.

- Imposes little on the cross-sectional returns.
- Does assume no correlation across time in returns.
- Equivalent to certain GMM specifications under these assumptions.

Fama-Macbeth estimation

1. Estimate β_t.
   For each security, i, estimate the time-series of β^i_t. This could be done for each t using a rolling window or other methods. (If using a constant β just run the usual time-series regression for each security.)

$$\tilde{r}^i_t = \alpha^i + \beta^{i,z}_t \; z_t + \epsilon^i_t$$

2. Estimate λ, υ.[1]
   For each t, estimate a cross-sectional regression to obtain λ_t and estimates of the N pricing errors, υ^i_t.

$$\tilde{r}^i_t = \beta^{i,z}_t \lambda_t + \upsilon^i_t$$

[1] Could include an intercept here, though LFM implies no intercept.

Illustration of time and cross regressions

Use sample means of the estimates:

$$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T}\lambda_t, \qquad \hat{\upsilon}^i = \frac{1}{T}\sum_{t=1}^{T}\upsilon_t^i$$

- This allowed flexible model for $\beta_t^{i,z}$.
- Running t cross-sectional regressions allowed t (unrelated) estimates λ_t and υ_t.

Fama-MacBeth standard errors

Get standard errors of the estimates by using Law of Large Numbers for the sample means, $\hat{\lambda}$ and $\hat{\upsilon}$.

$$
\begin{aligned}
s.e.(\hat{\lambda}) &= \frac{1}{\sqrt{T}}\sigma_\lambda \\
&= \frac{1}{T}\sqrt{\sum_{t=1}^{T}\left(\lambda_t - \hat{\lambda}\right)^2}
\end{aligned}
$$

- These standard errors correct for cross-sectional correlation.
- If there is no time-series correlation in the OLS errors, then the Fama-Macbeth standard errors will equal the GMM errors.

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Beyond Fama-MacBeth

The Fama-MacBeth, two-pass, regression approach is very popular to incorporate dynamic betas.[2]

- It is easy to implement.
- It is (relatively!) easy to understand.
- It gives reasonable estimates of the standard errors.

If we want to calculate more precise standard errors, we could easily use the Generalized Method of Moments (GMM).

- GMM would account for any serial correlation.
- GMM would account for the imprecision of the first-stage (time-series) estimates.

[2] Note that there would be no point of using Fama-MacBeth if we are using full-sample time-series betas. This will just give us the usual cross-sectional estimates.

References

- Back, Kerry. *Asset Pricing and Portfolio Choice Theory.* 2010. Chapter 6.
- Bodie, Kane, and Marcus. *Investments.* 2011. Chapters 9 and 10.
- Cochrane. *Discount Rates*. Journal of Finance. August 2011.
- Frazzini, Adrea and Lasse Pedersen. *Betting Against Beta.* Working Paper. October 2011.